

Entanglement Entropy on the Cayley Tree

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The properties of the entanglement entropy (EE) of a clean Cayley tree (CT) are studied. The EE shows a completely different behaviour depending on the way the CT is partitioned into two regions and whether we consider the ground-state or highly excited many-particle wave function. The ground-state EE increases logarithmically as function of number of generation if a single branch is pruned off the tree, while it grows exponentially if the region around the root is trimmed. On the other hand, in both cases the highly excited states' EE grows exponentially. Implications of these results to general graphs and disordered systems are shortly discussed.

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I. INTRODUCTION

In recent years there has been a renewed interest^{1,2} in the problem of Anderson localization on the Cayley tree (CT) (in the infinite limit known as a Bethe lattice). This interest is mainly motivated by the connection between the many body localization phenomenon³ and the CT. Many body localization may be viewed as a localization problem in Fock space, where the coupling between states due to electron-electron interactions resembles the CT⁴.

In the context of the Anderson localization on the CT⁵, one can envisage that the entanglement entropy (EE) will play an important role in clarifying the CT localization properties. In this EE could join other methods such as level spacing statistics for the CT systems^{2,6-8}, and for many-body interacting systems believed to map on an effective CT in Fock space⁹⁻¹⁵. Prior to addressing the challenging question of the EE on a disordered one must clarify the behavior of the EE for a clean CT. This is our main goal in this paper.

The EE is a measure of the entanglement between two regions, A and B, of a system which is in some pure state. This measure is given by the von-Neumann entropy of the reduced density matrix, ρ_A , of region A:

$$S_A = -\text{Tr}(\rho_A \ln \rho_A). \quad (1)$$

It has been shown that the EE is a very useful measure to locate and analyze quantum phase transitions¹⁶⁻²². The ground state EE typically scales like the *boundary area* of the region^{16,17}. Thus, for one dimensional systems, one would expect that the EE will be constant. This is correct for gaped systems, but for metallic systems there is a logarithmic correction, and EE grows like $\ln(L_A)$, where L_A is the size of the sub-system A. For insulators the EE should not depend on the region's size. Therefore for disordered systems the EE can give an indication for the localization length ξ ; we expect the EE to grow logarithmically for $\xi \gg L_A$, and to saturate for $\xi \ll L_A$ ²⁰.

Applying this picture to the CT is not straight forward. Unlike the situation for regular d dimensional systems, where for any regular simply connected area, the boundary area is proportional to L_A^{d-1} , for the CT things are more complicated. One expects that a region which

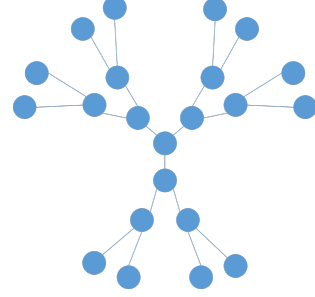


FIG. 1: A Cayley tree with coordination number $Z = 3$ and $N = 4$ generations

is connected to the rest of the graph only at one point (a branch of the tree), will show a different EE than a region centered at the root of the tree where the boundary area grows as $\sim C^{L_A}$ (where C is related to the coordination number of the tree). In this paper we aim to calculate the EE for a clean CT as a stepping stone towards investigating the EE in a disordered CT. We shall pay special attention to the different behavior for different regions and the physical meaning of these results.

II. CAYLEY TREE

A CT is²³ a simple connected undirected graph, with no closed loops. A CT of coordination number Z and N generations is described as follows. There is a root vertex, which we denote (1). The root is linked to Z vertexes belonging to the 2nd generation, which we denote (2), (3), ..., $(Z + 1)$. Each of those is linked to another $Z - 1$ vertexes on the 3rd generation, so that (2) is related to $(Z + 2)$, ..., $(Z^2 + 1)$, and so on. Every vertex is linked to Z others, except those belonging to the last generation, which are linked to only one vertex each. Unlike the *Bethe Lattice*, CT is finite, it's boundary being a non-negligible part of the entire tree: the total number of vertexes in the tree is $1 + Z \frac{(Z-1)^{N-1}-1}{Z-2}$, while the last generation contains $Z(Z-1)^{N-2}$ vertexes. Therefore, even for $N \rightarrow \infty$ the boundary is important, as is well known in the context of localization on CT^{6,24}.

Another noteworthy property of the CT is its lack of dimension - since the number of vertexes grows exponentially with N , we cannot define the dimension of this system. On the other hand, any two vertexes are connected by a single path, what gives the CT a one-dimensional character. Both of these features are reflected in the scaling of the EE we obtained.

III. CT EIGENSTATES

First we solve the tight-binding Hamiltonian for a clean CT. The Hamiltonian can be written, using the notation introduced previously, in the form

$$\hat{H} = -t \sum_{i=2}^{Z+1} (\hat{c}_1^\dagger \hat{c}_i + h.c.) - t \sum_{g=2}^{N-1} \sum_{i=1}^{Z(Z-1)^{g-2}} \sum_{j=1}^{Z-1} (\hat{c}_{p_g+i-1}^\dagger \hat{c}_{p_{g+1}+(Z-1)(i-1)+(j-1)} + h.c.) \quad (2)$$

where $\hat{c}_i^{(\dagger)}$ is an annihilation (creation) operator on vertex i , and $p_g = 2 + Z \frac{(Z-1)^{g-2}-1}{Z-2}$ is the index of the first vertex of the g 's generation. It will be convenient to define a new basis, using the operators $\{\hat{b}^{(\dagger)}\}$, defined as follows. We find an orthonormal basis in the 2nd generation, $\{\hat{b}_2^{2,\nu(\dagger)}\}_{\nu=1}^Z$, and for each such operator we define its successors in the following generations $\{\hat{b}_j^{2,\nu(\dagger)}\}_{j=3}^N$ as a reflection of $\hat{b}_2^{2,\nu(\dagger)}$ in the j 's generation. For example, if $\hat{b}_2^{2,\nu} = \frac{1}{\sqrt{2}}(\hat{c}_2 - \hat{c}_3)$, then $\hat{b}_3^{2,\nu} = \frac{1}{\sqrt{2(Z-1)}}(\sum_{i=Z+2}^{2Z} \hat{c}_i - \sum_{j=2Z+1}^{3Z-1} \hat{c}_j)$ etc. Similarly we find a basis in every generation, composed of operators other than the successors of the previous generations, and define its successors in the next generations. We denote each such operator by $\hat{b}_j^{\Gamma,\nu(\dagger)}$, where j is the generation in which this operator acts, Γ is the generation in which its first fore-father acts, and ν is its fore-father's index inside the Γ 's generation. Among these operators there is one important group that originates from the root, which we shall denote $\hat{b}_j^{symmetric}$ or \hat{b}_j^s , ($j = 1, \dots, N$). They are symmetric in the sense that they treat each generation as one unit, *i.e.*

$$\hat{b}_j^s = \begin{cases} \hat{c}_1 & \text{for } j = 1 \\ \frac{1}{\sqrt{Z(Z-1)^{j-2}}} \sum_{i=p_j}^{p_{j+1}-1} \hat{c}_i & \text{for } N \geq j > 1 \end{cases} \quad (3)$$

Using this representation, the Hamiltonian becomes

$$\begin{aligned} \hat{H} = & \hat{H}_s - t\sqrt{Z-1} \left[\sum_{j=2}^{N-1} \sum_{\nu=1}^{Z-1} (\hat{b}_j^{2,\nu\dagger} \hat{b}_{j+1}^{2,\nu} + h.c.) + \right. \\ & \left. + \sum_{j=3}^{N-1} \sum_{\Gamma=3}^j \sum_{\nu=1}^{Z(Z-2)(Z-1)^{\Gamma-3}} (\hat{b}_j^{\Gamma,\nu\dagger} \hat{b}_{j+1}^{\Gamma,\nu} + h.c.) \right] \quad (4) \end{aligned}$$

where the symmetric term

$$\hat{H}_s = -t\sqrt{Z}(\hat{b}_1^{s\dagger} \hat{b}_2^s + h.c.) - t\sqrt{Z-1} \sum_{j=2}^{N-1} (\hat{b}_j^{s\dagger} \hat{b}_{j+1}^s + h.c.). \quad (5)$$

Thus all terms, except for \hat{H}_s , are a sum of independent one-dimensional lattices. One should note the exponentially increasing number of such lattices. The solution is given by the transformation

$$\hat{d}_k^{\Gamma,\nu} = \sqrt{\frac{2}{N-\Gamma+2}} \sum_{j=\Gamma}^N \sin \left[\frac{(j-\Gamma+1)k\pi}{N-\Gamma+2} \right] \hat{b}_j^{\Gamma,\nu} \quad (6)$$

so that

$$\begin{aligned} \hat{H} = & \hat{H}_s - \\ & -t\sqrt{Z-1} \sum_{\Gamma=2}^{N-1} \sum_{\nu} \sum_{k=1}^{N-\Gamma+1} 2 \cos \left(\frac{k\pi}{N-\Gamma+2} \right) \hat{d}_k^{\Gamma,\nu\dagger} \hat{d}_k^{\Gamma,\nu} \end{aligned} \quad (7)$$

and the spectrum is

$$\varepsilon_{k,\Gamma} = -2t\sqrt{Z-1} \cos \left(\frac{k\pi}{N-\Gamma+2} \right). \quad (8)$$

For the symmetric term (5) a similar solution was obtained by Chen *et al.*²⁵, including the boundary condition for finite N .

IV. ENTANGLEMENT ENTROPY

In order to calculate the EE we use the relation²⁶

$$S = \sum_l -\lambda_l \ln \lambda_l - (1 - \lambda_l) \ln(1 - \lambda_l) \quad (9)$$

where $\{\lambda_l\}$ are the eigenvalues of the correlation matrix in region A . For a one-particle state, the correlation can be written in the form $\hat{C} = |\psi_L\rangle \langle \psi_L|$, where $|\psi_L\rangle$ is the eigenstate in region A . Thus one eigenvector is $|\psi_L\rangle$, with corresponding eigenvalue $\lambda = \langle \psi_L | \psi_L \rangle$. Since \hat{C} is Hermitian, any other eigenvector is orthogonal to $|\psi_L\rangle$, and its corresponding eigenvalue therefore vanishes, so that the EE is simply $S = -\lambda \ln \lambda - (1 - \lambda) \ln(1 - \lambda)$. We have

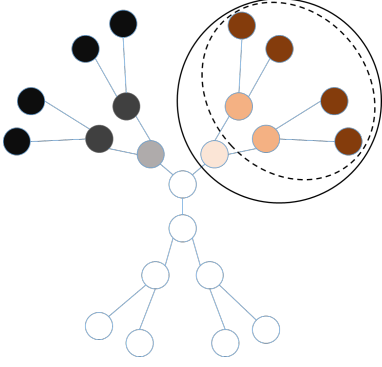


FIG. 2: A typical eigenstate in a CT with $Z = 3$, when we take one branch (solid line) or a part of it (dashed) to be the subsystem. (Color online)

calculated the EE for several representative one-particle states in a CT of $Z = 3$. A typical state is:

$$|\psi_1^{2,1}\rangle = \hat{d}_1^{2,1\dagger} |\emptyset\rangle = \sum_{j=2}^N \frac{1}{\sqrt{2^{j-2}N}} \sin\left[\frac{(j-1)\pi}{N}\right] \cdot \left[\sum_{i=p_j}^{p_j+2^{j-2}-1} \hat{c}_i^\dagger - \sum_{l=p_j+2^{j-2}}^{p_j+2^{j-1}-1} \hat{c}_l^\dagger \right] |\emptyset\rangle \quad (10)$$

which resides on two of the branches. For this state, we let one branch be region A (see Fig. 2), and obtain $\lambda = \langle \psi_L | \psi_L \rangle = \frac{1}{2}$, and $S = -\ln \frac{1}{2} = \ln 2$, regardless of the number of generations. Another possible partition is to define region A as a part of one branch (see in Fig. 2). The eigenvalue λ_M for this partition where region A is defined as the last M generations of one branch is

$$\lambda_M = \frac{M}{2N} + \frac{1}{4N} \left(1 - \frac{\sin\left[\frac{\pi}{N}(2M+1)\right]}{\sin\left(\frac{\pi}{N}\right)} \right),$$

which, in the appropriate limit, is proportional to the relative number of generations in region A.

For the more challenging case of a many-particle state, the correlation matrix becomes more complicated, so we diagonalize it numerically. The many particle state can be written as

$$|\psi\rangle = \prod_{\Gamma, \nu, k} \hat{d}_k^{\Gamma, \nu\dagger} |\emptyset\rangle, \quad (11)$$

where Γ, ν, k are the indices of the occupied single particle states, leading to a correlation matrix

$$C_{n,m} = \langle \hat{c}_n^\dagger \hat{c}_m \rangle = \langle \emptyset | \prod_{\Gamma, \nu, k} \hat{d}_k^{\Gamma, \nu\dagger} \hat{c}_n^\dagger \hat{c}_m \prod_{\Gamma', \nu', k'} \hat{d}_{k'}^{\Gamma', \nu'\dagger} |\emptyset\rangle. \quad (12)$$

Let us first consider the ground state EE of a single branch of the CT (see in Fig. 5) for the half-filled case. Since a branch is connected to the rest of the system only

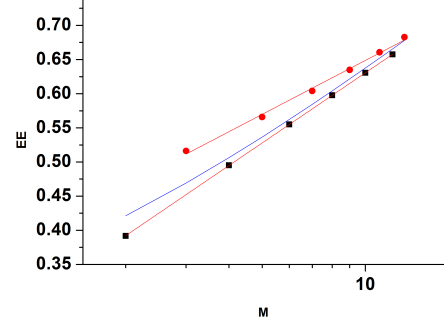


FIG. 3: Ground state EE between one branch and the rest for $Z = 3$ as function of M the number of generations in the subsystem. Here $M = N - 1$. The black squares represent an even number of generations and the red circles represent odd M 's, separated in order to emphasize the alternating term. The red lines correspond to $0.15 \ln(M) + 0.29$ for even M 's and $0.11 \ln(M) + 0.39$ for odd M 's. The blue line in between is $\frac{1}{6} \ln(N) + 0.24$, which corresponds to the asymptotic behavior of the EE.

at one point, one would expect from the area law that the EE will be constant. Nevertheless, since the CT has no gap, which is similar to the situation in metallic 1D system, a logarithmic correction is expected. Indeed, we find a logarithmic dependence for the CT, but unlike the 1D system it does not depend on the size of the system, but rather on the number of generations N in the CT (or, equivalently, on the number of generations M in the subsystem, where in this case $M = N - 1$). Specifically, for $Z = 3$, $S_A \sim \frac{1}{6} \ln N + (-1)^N \gamma(N)$, where $(-1)^N \gamma(N)$ is a decaying alternating term typical to systems with open boundary conditions²⁷ (Fig. 3). The pre-factor $\frac{1}{6}$ of the logarithmic term is reminiscence of the behavior of an open boundary 1D system, where if one bisects a system of length L in the middle the EE grows as $\frac{1}{6} \ln L$ ^{16,17}. The fact that for the EE on a CT the number of generations N plays the same role as the length L for a 1D system stems from the structure of the eigenstates of the CT demonstrated in Sec. III. It is interesting to point out that for the CT the ratio of the sites on a branch to the overall number of sites is nearly constant ($\sim 1/Z$) for any generation N larger than $O(1)$, again similar to the situation for the bisected 1D system.

On the other hand, for an excited state we find an exponential growth of the EE (Fig. 4), reflecting the growth in the number of sites per generation. This is expected since for an excited state the EE should obey a volume law. Nevertheless, this is in stark difference to the one dimensional case for which excited states EE grows linearly. Thus, for the EE of a branch of the CT there is a huge change between the ground state EE and the excited state EE not seen in more standard models. This may lead to a huge influence of localization on low-lying excitations²².

A similar form of dissection is to define a site residing

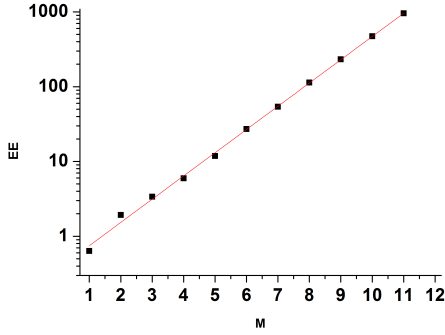


FIG. 4: EE between one branch and the rest of the CT with $Z = 3$ for a highly excited state represented by the black squares. The EE grows as $0.39 \exp(0.71M) - 0.65 \approx 0.39 \times 2^{1.02M} - 0.65$ (red line) since for an excited state the EE is proportional to the volume.

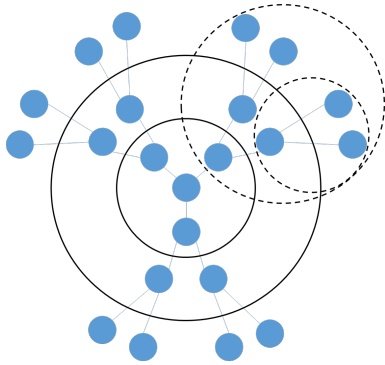


FIG. 5: Different ways to partition the CT: taking M generations around the root to be region A (solid circles) or taking a branch (big dashed circle) or a sub-branch (small dashed circle).

M generations away from the boundary and all the sites connecting it to the boundary as region A, so that A is connected to the rest of the tree only at one point (see in Fig. 5), just like in the previous case. As expected, we get the logarithmic dependence for this case also (Fig. 6).

A different obvious way to cut the CT into two different regions is to define the root and the sites belonging to the first M generations as region A, and the rest of the sites belonging to the higher generations as region B (Fig. 5). In this case the area of contact between the two regions is proportional to $Z(Z-1)^{M-2}$ for $M \geq 2$, and therefore according to the area law we expect the ground state EE to be exponentially dependent on M . Indeed, in this case the EE behaves as $S_A \sim \exp(\alpha M)$ (see Fig. 7). Since for such a dissection the volume (number of sites) within region A is equal to $1 + Z \frac{(Z-1)^{M-1} - 1}{Z-2} \sim (Z-1)^{M-1}$, i.e., grows exponentially with M . Also for a highly excited state we find the EE to follow $S_A \sim \exp(\alpha' M)$, as expected. Thus, for this form of dissection there is no qualitative difference between the EE of the ground state

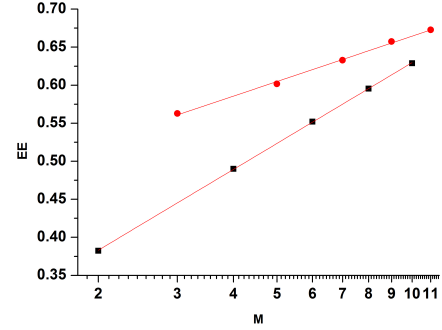


FIG. 6: EE when we cut the CT to sub-branches with M generations, where the total number of generations in the tree is $N = 13$. The black squares represent even M 's, with the red line corresponding to $0.15 \ln(M) + 0.28$, and the red circles are for odd M 's, with the line corresponding to $0.086 \ln(M) + 0.47$.

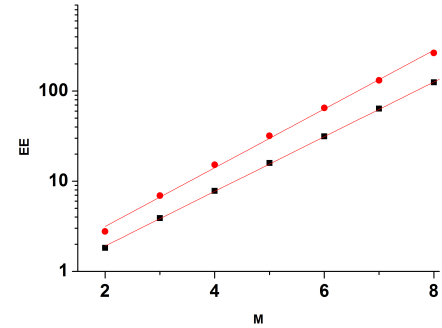


FIG. 7: EE between the first M generations around the root (including the root itself) and the rest of the CT with $Z = 3$ for the ground state (black squares) and for a highly excited state (red circles). In both cases the EE grows exponentially (the lines, corresponding to $0.55 \exp(0.68M) - 0.44 \approx 0.55 \times 2^{0.98M} - 0.44$ and $1.04 \exp(0.69M) - 1.39 \approx 1.04 \times 2^M - 1.39$, respectively), since there is no qualitative difference in the growth as function of M between the interface area and the volume.

and the behavior of an excited state as function of M .

V. DISCUSSION

In this paper we calculated the EE for the ground state and excited states of a clean CT. Utilizing an exact expression for the single-particle eigenfunctions of the CT developed in this paper in order to obtain a correlation matrix from which the EE of a many-particle state can be calculated, it has been demonstrated that the EE may show dramatic differences depending on the way the CT is partitioned into two regions. If the CT is dissected into a branch connected in a single point to the CT, a logarithmic dependence on the number of generations in

the branch is found, which reflects the one dimensional character of the states of the CT. On the other hand, excited states on the branch show a completely different EE behavior, namely an exponential dependence on the number of generations in the branch. This logarithmic to exponential change in the entanglement between the ground state and the excited state is unique to the entanglement on the CT. On the other hand, the ground state EE shows a completely different behavior for a different sort of partition - where the root and the first M generations are considered as one region. In this case an exponential dependence on M is found, which reflects the growth in the number of sites on the surface. In contrast to dissecting by branch, here the excited states EE (which remains the volume law) does not show a dramatic difference when compared to the ground state EE.

Thus, how one defines the separation into two regions has a crucial influence on the behavior of the EE. These large fluctuations in the EE could become even stronger for other types of complex (random) networks, such as small world graphs, Erdős-Rényi graphs and scale free

networks²⁸.

These behaviors predict a strong influence of disorder on the EE of a CT. It has been shown²⁰ that the localization length in 1D may be inferred from the saturation of the EE on the length scale of the localization length. For the CT one expects that for a region around the root and the first M generations where the increase in the EE is exponential this tool will be more sensitive than for the 1D system where the growth is logarithmic. It will be also very interesting to compare the effect of the localization on the EE of a branch. Will it be similar to the 1D case? Another question is whether the non ergodic phase predicted for the CT² may be detected in the behavior of the EE.

Acknowledgments

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